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Laminar Boundary Layers Developed within Unsteady Expansion and Compression Waves

J. GORDON HALL*
State University of New York at Buffalo, Buffalo, N. Y.

The paper analyzes unsteady laminar wall boundary layers formed within centered compression or expansion waves traveling into gas at rest, and also the resultant displacement effects on inviscid flow in a tube. For the boundary layer, a previous analysis by Cohen is generalized to include both compression and expansion flows and allow for wall surface temperature change as occurs with finite wall thermal conductivity. Solutions obtained by coordinate expansions show that at higher pressures the wall surface temperature change significantly reduces surface heat transfer, outweighing higher order effects of the previous analysis for wall temperature constant. For the displacement effect, the method of characteristics is used to obtain linearized solutions. Characteristic features of the boundary layer and its displacement effect on inviscid tube flow are compared for compression and expansion wave flows. For either type of wave, boundary-layer displacement acts to diminish the basic inviscid-flow changes produced by the wave.

Nomenclature

a = gas sound speed
A_0, A_1, A_2 = coefficients for v_e, Eq. (24)
b_i = coefficients for T_W, Eq. (15)
C_B = wall specific heat
C_p, C_v = gas specific heats (const)
D = hydraulic diameter = 4 × cross section area/perimeter
erfc(z) = $[2/(\pi)^{1/2}]\int_z^\infty e^{-s^2}\,dS$
f_0, f_1, f_2 = constants in Eq. (24)
F = defined by Eq. (8)
g_0, g_1, g_2 = constants in Eq. (24)
G = defined by Eq. (9)
h = specific enthalpy $C_p T$
H = defined by Eq. (12)
k = coefficient of thermal conductivity
K = $(\rho_B C_B k_B)/(\rho_0 C_p k_0)$
p = gas pressure
q = heat-transfer rate, $-k\,\partial T/\partial y$
$\bar{q}$ = dimensionless heat-transfer rate defined by Eq. (22)
s = dimensionless independent variable, $x/(a_0 t)$
t = time measured relative to wave focus, $t' - t_0'$
t' = time measured relative to expansion wave origin
t_0' = time of wave focus, zero for expansion wave
T = gas temperature
ΔT = difference between stream and wall surface temperature, $T_e - T_W$
u = flow velocity along x' outside of boundary layer in tube-fixed coordinate system x', t'
$\bar{u}$ = flow velocity along x in x, y, t coordinate system attached to wavehead
v = flow velocity along y normal to the wall
x = distance from wavehead, $x' + a_0 t'$
x' = distance from expansion wave origin
y = distance normal to wall from wall-gas interface
α = thermal diffusivity of wall, $k_B/(\rho_B C_B)$
β = $2/(\gamma+1) = 1 - \varepsilon$
γ = C_p/C_v, const
ε = $(\gamma-1)/(\gamma+1)$
η = $[a_0/(\nu_0 x)]^{1/2}\int_0 (\rho/\rho_0)\,dy$
$\bar{\eta}$ = $-y[a_0/(\alpha x)]^{1/2}$
θ = local temperature in the wall
μ = coefficient of viscosity of the gas
ν = kinematic viscosity of the gas, μ/ρ
ρ = mass density
σ = Prandtl number of the gas $\mu C_p/k$ = const
τ = shear stress $\mu\,\partial\bar{u}/\partial y$
$\bar{\tau}$ = dimensionless shear stress defined by Eq. (22)

Subscripts

B = denotes value for homogeneous wall material
e = denotes inviscid flow conditions in the absence of wall boundary layer
0 = denotes initial state of gas at rest ahead of wave
W = denotes conditions at gas-wall interface or $y = 0$

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* Professor, Department of Mechanical Engineering, and Director, Fluid and Thermal Sciences Laboratory. Member AIAA.

Introduction

THE present paper analyzes the unsteady laminar wall boundary layer formed within a finite expansion or compression wave traveling into a tube containing gas at rest, and the resulting displacement effects of the boundary layer on the unsteady inviscid flow in the tube interior. The type of flow in the expansion-wave case occurs, for example, in the high-pressure or driver section of a conventional shock tube after the diaphragm bursts (Fig. 1). A similar centered unsteady expansion-wave flow appears in the reservoir tube of a Ludweig or tube-type wind tunnel.[1] The type of flow in the compression-wave case might be similar to that generated by an accelerating piston (Fig. 2) such as employed in free-piston shock tubes[2] or isentropic compression tubes.[3] For both types of flow the extent

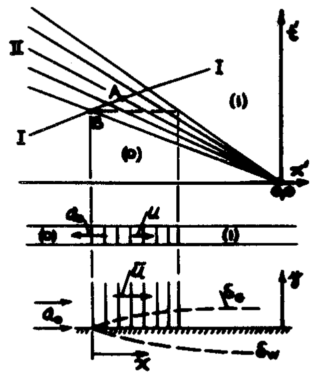

Fig. 1 Characteristics diagram and coordinates for expansion-wave flow. δ_G denotes gas boundary layer. δ_W denotes thermal layer in wall.

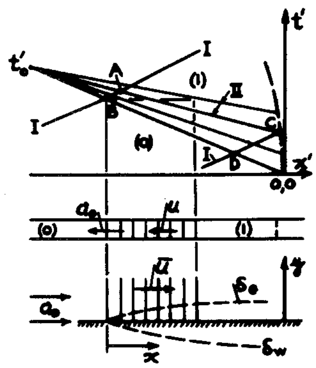

Fig. 2 Characteristics diagram and coordinates for compression-wave flow. δ_G denotes gas boundary layer, δ_W denotes thermal layer in wall.

of the wave region proper, through which the local flow properties change, varies with time. The wave thickness or extent increases with time for the expansion wave and decreases for the compression wave. This feature causes the induced wall boundary-layer flow to be inherently unsteady, in contrast to the analogous wall boundary layer formed behind a shock wave traveling into gas at rest.

The expansion-wave boundary layer first received attention in connection with the question of wall viscous effects in shock-tube flows.[4-6] In that connection Mirels[5] considered the laminar and turbulent boundary layers behind the expansion wave with the approximations of steady flow and a concentrated wave of zero thickness. The first analyses of the unsteady expansion-wave boundary layer appear to be those for laminar flow of Trimpi and Cohen[7] who used an integral method and Cohen[8] who used a coordinate expansion method with the differential equations. About the same time, Chabai[9] measured wall heat-transfer in the driver section of a shock tube for both laminar and turbulent expansion-wave boundary layers. Subsequently, some theoretical analyses were done in connection with expansion-wave boundary layers in tube-type wind tunnels. The latter include calculations by Becker[10] of the unsteady turbulent wall boundary layer. A comprehensive review of the earlier studies of both shock-tube and tube-type wind-tunnel boundary layers has been given by Becker.[10] Shock-tube wall boundary layers for laminar flow, including the expansion wave flow, are discussed in some detail in Stewartson's book.[11]

The present paper analyzes three features of such unsteady laminar boundary layers not previously treated. The first of these is the effect of wall surface temperature change which occurs with finite wall thermal conductivity. Previous analyses[7,8,10] neglect the wall temperature change, which avoids consideration of heat flow in the wall but requires infinite wall conductivity. Although typically small changes in wall temperature occur, neglect of such changes introduces significant error in predicted heat transfer for some experimental conditions of interest.

By means of coordinate expansions the present analysis solves the wall heat conduction equation in boundary layer form together with the gas boundary-layer equations. The zero order solution, which for the gas boundary layer is identical to that of Cohen,[8] along with the new surface boundary condition then determines the first-order wall temperature distribution and corresponding heat transfer.

The second feature considered herein is the boundary layer formed within centered compression waves analogous to expansion waves. To accommodate both types of flows the problem is described initially in terms of time measured relative to the wave focal point and a streamwise distance measured relative to the wave head, in addition to an appropriate normal coordinate. This approach provides a unified formulation, including wall temperature variation, from which the coordinate expansion solutions may be interpreted for either expansion or compression type flows. The previous solution of Cohen[8] for expansion-wave flow with constant wall temperature is used to construct the corresponding compression wave solution.

The third new feature treated is the local displacement effect of such unsteady boundary layers on the interior inviscid flow in a tube. The approach here uses the method of characteristics with the boundary-layer influence represented as a mass source or sink in terms of the normal velocity at its outer edge. Linearized solutions for the displacement effects are obtained for the particular case of laminar wall boundary layers.

Laminar Wall Boundary Layer in Finite Expansion and Compression Waves

Governing Equations and Boundary Conditions

Figures 1 and 2 illustrate the flowfield and the coordinate systems used. The expansion or compression wave travels into gas at rest, the wave head moving at the constant sound speed a_0, and is assumed to be a focused or centered simple wave in both cases. The gas is assumed perfect, and the (thin) boundary layer on the wall is assumed to be two-dimensional and unsteady as well as laminar and compressible. In a coordinate system attached to the wave head, the customary boundary-layer approximations then give the following equations:

Continuity:

$$\partial\rho/\partial t+(\partial/\partial x)(\rho\bar{u})+(\partial/\partial y)(\rho v)=0 \tag{1}$$

Momentum:

$$\rho\frac{D\bar{u}}{Dt}=\rho\left(\frac{\partial\bar{u}}{\partial t}+\bar{u}\frac{\partial\bar{u}}{\partial x}+v\frac{\partial\bar{u}}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial\bar{u}}{\partial y}\right) \tag{2}$$

Energy

$$\rho\frac{Dh}{Dt}=\frac{\partial p}{\partial t}+\bar{u}\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\frac{\mu}{\sigma}\frac{\partial h}{\partial y}\right)+\mu\left(\frac{\partial\bar{u}}{\partial y}\right)^2 \tag{3}$$

as well as $\partial p/\partial y=0$, or $p=p_e=p_e(x,t)$. In these coordinates x is streamwise distance measured from the stationary wave head and y is distance normal to the wall. The gas approaches the wave head at uniform velocity a_0 which is also the velocity of the wall. The time $t=t'-t_0'$ is measured from the time t_0' at which the wave is focused. For the expansion wave $t_0'=0$, and $t=t'$. For the compression wave, $t'<t_0'$ so that t is actually a negative quantity in that case.

The boundary conditions for Eqs. (1–3) are as follows:

At $x=0$:

$$\bar{u}=a_0,\quad T=T_0,\quad \rho=\rho_0,\quad p=p_e=p_0.$$

For $y\to\infty$ (i.e., in the inviscid outer flow):

$$\bar{u}=\bar{u}_e(x,t),\quad T=T_e(x,t),\quad p=p_e(x,t)$$

At $y = 0$:

$$\bar{u} = a_0, \quad v = 0, \quad T = T_W(x,t), \quad \rho = \rho_W(x,t), \quad p = p_e(x,t).$$

For the case of centered waves considered, the outer inviscid flow quantities are determined in terms of $s = x/(a_0 t)$ and γ by simple wave relations to be given later which become of identical form for compression and expansion waves.

The unknown wall surface temperature $T_W(x,t)$ is initially decreased from its (usual) initial value of T_0 for an expansion wave and increased over T_0 for a compression wave because of the heat transfer to or from the gas. A consistent analysis of the problem then requires the simultaneous solution of the gas boundary layer and the thermal boundary layer developed within the wall, with matching of the gas and wall heat-transfer rates at the surface. $T_W(x,t)$ is then obtained as part of the over-all solution.

The distribution of temperature θ in the wall ($y < 0$ and assumed homogeneous) is governed by the heat conduction equation which is approximated by

$$\partial\theta/\partial t + a_0\,\partial\theta/\partial x = \alpha\,\partial^2\theta/\partial y^2 \tag{4}$$

where $\alpha = k_B/(\rho_B C_B)$ is the thermal diffusivity of the wall material. In Eq. (4), $\partial^2\theta/\partial x^2$ has been neglected compared to $\partial^2\theta/\partial y^2$, consistent with the gas boundary-layer approximations. The boundary conditions for Eq. (4) are taken to be $\theta = T_0$ at $x = 0$ and at $y \to -\infty$, and $\theta = T_W(x,t)$ at $y = 0$.

The final boundary condition required is that for matching the two heat-transfer rates at $y = 0$, i.e.,

$$q_W = -k_W[(\partial T/\partial y)]_{y=0} = -k_B[(\partial\theta/\partial y)]_{y=0} \tag{5}$$

Of course the preceding model for the wall thermal layer requires a wall which is not only homogeneous but also adequately thick. In practice this requirement is usually met. The ratio of the wall to the gas thermal layer thicknesses is $(\alpha/v_0)^{1/2}$. For air at 70°F and 1 atm pressure this ratio is about unity for a steel wall and about 0.2 for glass.

Reduction of Equations

The number of independent variables in the gas boundary layer Eqs. (1–3) can be reduced from three to two if the equations are transformed in terms of the streamwise variable $s = x/(a_0 t)$ and a normal coordinate (η) of the usual boundary-layer type, i.e., proportional to $x^{-1/2}\int \rho\, dy$. A convenient definition of η is

$$\eta = [(a_0/v_0 x)]^{1/2}\int_0 (\rho/\rho_0)\, dy \tag{6}$$

The absolute magnitude of the variable s is the ratio of the distance x from the wave head to the distance $|a_0 t| = |a_0(t' - t_0')|$ of the wave head from the focal point. Thus for the expansion wave, s varies from 0 at the wave head to 1 at the origin $t' = 0$, $x' = 0$. For the compression wave, s is negative and approaches $-\infty$ for finite x as the wave head approaches the focal point ($t' \to t_0'$).

For centered waves, the outer inviscid flow quantities may now be expressed in terms of s by the following simple wave relations:

$$\bar{u}/a_0 = 1 + \beta s \tag{7a}$$

$$(p_e/p_0)^{\varepsilon/(\gamma\beta)} = (\rho_e/\rho_0)^{\varepsilon/\beta} = (T_e/T_0)^{1/2} = 1 - \varepsilon s \tag{7b}$$

$$\beta = 2/(\gamma + 1) \qquad \varepsilon = (\gamma - 1)/(\gamma + 1) \tag{7c}$$

where s is thus positive for expansion waves and negative for compression waves.

In accord with the preceding transformation and outer flow conditions, it is convenient to express the velocity component $\bar{u}$ and temperature T in terms of two functions $F(s,\eta)$ and $G(s,\eta)$ as follows:

$$\bar{u}/a_0 = 1 + \beta s\,\partial F/\partial\eta \tag{8}$$

$$T = T_W + (T_e - T_W)G \tag{9}$$

where $T_W = T_W(s)$. Assuming that the local viscosity μ is proportional to T, so that $\rho\mu = \rho_e\mu_e = \rho_0\mu_0 p_e/p_0$, Eqs. (1–3) can then be transformed to give the following two equations involving the unknown functions $F(s,\eta)$, $G(s,\eta)$ and $T_W(s)$.

$$\frac{p_e}{p_0}F_{\eta\eta\eta} + \left(\frac{\eta}{2} + \frac{3}{2}\beta sF + \beta s^2 F_s\right)F_{\eta\eta} - (sF_{s\eta} + F_\eta)\times$$
$$(1 - s + \beta sF_\eta) - \frac{dp_e/ds}{\gamma\beta T_0 p_e}(T_W + \Delta TG) = 0 \tag{10}$$

$$\frac{p_e}{\gamma p_0}G_{\eta\eta} + \left(\frac{\eta}{2} + \frac{3}{2}\beta sF + \beta s^2 F_s\right)G_\eta - s\left(\frac{1 - s + \beta sF_\eta}{\Delta T}\right)\times$$
$$\left[(T_W + \Delta TG)_s - \frac{2\varepsilon\, dp_e/ds}{\beta\gamma p_e}(T_W + \Delta TG)\right] +$$
$$\frac{2\varepsilon\beta T_0 p_e}{p_0\Delta T}s^2 F_{\eta\eta}{}^2 = 0 \tag{11}$$

Here the subscripts s and η denote partial derivatives, and $F(s,0)$ has been set equal to zero without loss of generality.

For the thermal boundary layer in the wall, because of the boundary-layer approximation made in the wall heat conduction equation a similar transformation in terms of s and a second normal coordinate $\bar{\eta}$ defined as $\bar{\eta} = -y[a_0/(\alpha x)]^{1/2}$ now reduces Eq. (4) to one involving s and $\bar{\eta}$ only. Introducing a function $H(s,\bar{\eta})$ defined by

$$\theta = T_W + (T_0 - T_W)H \tag{12}$$

the transformed Eq. (4) then becomes

$$H_{\bar{\eta}\bar{\eta}} + (\bar{\eta}/2)H_{\bar{\eta}} - s(1-s)\{[(1-H)(dT_W/ds)/(T_0 - T_W) + H_s]\} = 0 \tag{13}$$

Finally, transformation of the boundary condition for matching the surface heat-transfer rates, Eq. (5), gives

$$q_W = -\frac{p_e k_0}{p_0 v_0{}^{1/2}}(T_e - T_W)(G_\eta)_{\eta=0}$$
$$= \frac{k_B}{\alpha^{1/2}}(T_0 - T_W)(H_{\bar{\eta}})_{\bar{\eta}=0} \tag{14}$$

Thus the problem for both centered expansion and compression waves with varying surface temperature is reduced to the solution of Eqs. (10), (11), (13), and (14) for the four functions $F(s,\eta)$, $G(s,\eta)$, $H(s,\bar{\eta})$, $T_W(s)$.

Method of Solution

It is now assumed that for small values of s, solutions to the preceding equations satisfying the boundary conditions given can be obtained by coordinate expansions in s of the form

$$Y = Y_0 + Y_1 s + Y_2 s^2 + \ldots$$
$$T_W = T_0(b_0 + b_1 s + b_2 s^2 + \ldots) \tag{15}$$

where the coefficients Y_i represent F_i, G_i or H_i and depend on η or $\bar{\eta}$ only, and the coefficients b_i are constants to be determined ($b_0 = 1$). The substitution of relations (15) into Eqs. (10, (11), and (13) gives a sequence of linear ordinary differential equations for F_i, G_i and H_i which have the following form.

$$F_i''' + (\eta/2)F_i'' - (1+i)F_i' = \bar{F}_i(F_{i-1}, G_{i-1}, b_i) \tag{16a}$$

$$(1/\sigma)G_i'' + (\eta/2)G_i' - (1+i)G_i = \bar{G}_i(F_{i-1}, G_{i-1}, b_i, b_{i+1}) \tag{16b}$$

$$H_i'' + (\bar{\eta}/2)H_i' - (1+i)H_i = \bar{H}_i(H_{i-1}, b_i, b_{i+1}) \tag{16c}$$

Here the right sides equal -1 for $i = 0$, and for $i = 1, 2, \ldots$ depend on the lower order functions F_{i-1}, G_{i-1} and H_{i-1}, and on the constants b of order i and $i+1$. The constants b_i are related to $G_{i-1}'(0)$ and $H_{i-1}'(0)$ by a corresponding set of algebraic equations obtained from Eq. (14) for surface heat transfer, which becomes

$$q_W = \frac{p_e k_0 T_0 s}{p_0}\left(\frac{a_0}{v_0 x}\right)^{1/2}[(b_1 + 2\varepsilon) + (b_2 - \varepsilon^2)s + b_3 s^2 + \ldots]\times$$
$$[G_0'(0) + sG_1'(0) + s^2 G_2'(0) + \ldots] = -k_B T_0 s\left(\frac{a_0}{\alpha x}\right)^{1/2}\times$$
$$[b_1 + b_2 s + b_3 s^2 + \ldots][H_0'(0) + sH_1'(0) + \ldots] \tag{17}$$

The boundary conditions on F_i', G_i, H_i are that these quantities vanish at η or $\bar{\eta}$ equal to 0 and ∞ with the exception that $F_0'(\infty) = G_0(\infty) = H_0(\infty) = 1$.

Equations (16) and (17) can be solved sequentially beginning with the lowest order equations for $i = 0$. A point to be noted is that the right sides of the equations for G_i and H_i depend implicitly on $G_i'(0)$ and $H_i'(0)$ through the coefficient b_{i+1} and Eq. (17), unlike the equation for F_i where $\bar{F}_i$ is explicitly known a priori in terms of previously determined lower order functions. This difficulty does not appear in the limiting case of infinite wall conductivity or constant wall temperature, where $b_i = 0$ for $i \geqq 1$. In that case, the preceding equations for F_i and G_i become the same as those given by Cohen[8] who treated the centered expansion wave for $k_B = \infty$, or constant T_W. Cohen obtained solutions for F_i and G_i up to $i = 2$, the solutions for $i = 1$ and 2 being numerically determined.

In the present study the primary interest was in determining the first-order change in wall temperature T_W, given by the coefficient b_1, and the associated effects on surface heat transfer. The solution of Eqs. (16) for variable T_W has accordingly been limited to the zero-order equations ($i = 0$) as only the functions G_0 and H_0 are needed to determine b_1. For the zero-order equations the right sides of Eqs. (16) have identical constant values of -1. The corresponding algebraic equation for b_1 is [from equating coefficients of s in Eq. (17)]

$$(k_0/v_0^{1/2})(b_1+2\varepsilon)G_0'(0) = -(k_B/\alpha^{1/2})b_1 H_0'(0) \tag{18}$$

Results for Variable T_W

The zero-order forms of Eqs. (16) do not depend on T_W and therefore their solutions are identical to those given by Cohen[8] for the expansion wave with $k_B = \infty$, or constant T_W. The solutions for F_0', G_0 and H_0 may be exactly expressed as (for any values of γ and σ)

$$Y_0 = 1-(1+z^2/2)\,\mathrm{erfc}(z/2)+(z/\pi^{1/2})e^{-z^2/4} \tag{19}$$

where Y_0 represents F_0', G_0 or H_0 with corresponding arguments $z = \eta$, $\sigma^{1/2}\eta$, or $\bar{\eta}$, respectively.

The effect of finite wall conductivity k_B can now be obtained by evaluation of b_1 from Eq. (18) using values of $G_0'(0)$ and $H_0'(0)$ from Eq. (19) of $2(\sigma/\pi)^{1/2}$ and $2/\pi^{1/2}$, respectively. The value of b_1 so found can be expressed as

$$b_1 = -2\varepsilon/(1+K^{1/2}) \tag{20}$$

where $K = \rho_B C_B k_B/(\rho_0 C_p k_0)$. With this value of b_1 the leading terms of the solutions for T, T_W and the surface heat-transfer rate q_W for expansion ($s > 0$) and compression ($s < 0$) waves are then as follows.

$$T/T_0 = 1-2\varepsilon[G_0+(1-G_0)/(1+K^{1/2})]s+\ldots \tag{21a}$$

$$T_W/T_0 = 1-2\varepsilon s/(1+K^{1/2})s+\ldots \tag{21b}$$

$$q_W = [4\varepsilon k_0 T_0/(1+K^{1/2})](p_e/p_0)(\sigma a_0 K/\pi v_0 x)^{1/2}s+\ldots \tag{21c}$$

In q_W, the inviscid flow pressure p_e/p_0 has been left in exact form. The velocity field and the skin friction at the surface are unaffected by the wall conductivity in their leading terms.

The effect of finite K is to reduce the over-all temperature difference across the boundary layer and thus the surface heat transfer. The limits of K are zero for a perfectly insulating or adiabatic wall material, and infinity for a perfectly conducting wall. For $K = 0$, Eqs. (21) give $T = T_W = T_0(1-2\varepsilon s+\ldots)$ and $q_W = 0$ i.e., $T = T_e(s)$ everywhere in the gas boundary layer (viscous heating is a higher order effect in s). The other limit for $K \to \infty$ gives $T = T_0(1-2\varepsilon G_0 s+\ldots)$ and $T_W = T_0$, with the factor $K^{1/2}/(1+K^{1/2})$ in q_W becoming unity.

Finite values of K cover a wide range depending on the gas and its temperature and pressure, as well as on the wall material. As illustrative examples for the expansion-wave flow, one may consider gas conditions appropriate to a shock-tube driver gas or to gas in the reservoir tube of a tube-type wind tunnel. In those cases, a representative temperature T_0 would be 70°F and p_0 might range up to 100–1000 atm. Representative gases would be air, hydrogen, and helium, and wall materials of interest would be steel and glass. Glass is of particular interest in connection with its common use as the backing material for thin-film resistance thermometers used to measure local surface heat-transfer rates. At T_0 of 70°F and pressures p_0 of 10, 100, and 1000 atm, values of $K^{1/2}$ are then as given in Table 1.

Table 1 Values of $K^{1/2}$

	Steel/air	Glass/air	Steel/H_2	Glass/H_2
p_0 = 10 atm	676	72.5	260	27.9
= 100 atm	214	23.0	82.0	8.83
= 1000 atm	67.6	7.25	26.0	2.79

The values of $K^{1/2}$ in Table 1 taken in conjunction with Eqs. (21) indicate that finite thermal conductivity of the wall material can appreciably reduce surface heat transfer under certain conditions, particularly for glass inserts in the wall and higher gas pressures. The effect of finite K in the zero-order solution, or the leading terms of T and q_W given by Eqs. (21), can outweigh the importance of subsequent higher-order terms in the expansions in s. This point is illustrated by Fig. 3 where the leading term of $q_W t^{1/2}(\rho_0 a_0^2 v_0^{1/2}) = \bar{q}_W$ given by Eqs. (21) is compared for several values of K with the expansion-wave solution of Ref. 8 for $K = \infty$ which includes higher-order terms in $G(s,\eta)$ up to $G_2 s^2$. It is seen from Fig. 3 that the effect of finite values of K in the zero-order solution dominates over the higher order terms at small values of s.

It may be noted finally that the higher pressure conditions which increase the importance of finite K will also favor early boundary-layer transition. Thus the laminar boundary layer will generally be restricted to small values of s at such condition. In that case the leading term approximations of Eqs. (21) may provide an adequate solution without need for higher-order terms.

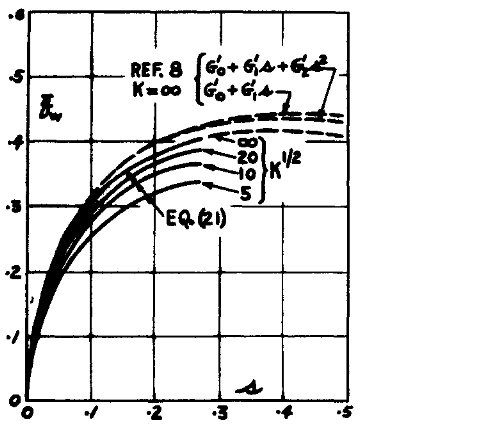

Fig. 3 Dimensionless surface heat-transfer rate from Eq. (21) for finite thermal conductivity of wall.

Results for Compression Waves

In the preceding formulation and results, the only difference between the two cases of expansion and compression-wave flows is in the sign of s which is negative for the latter case. Thus for compression flows the terms of odd order in the series expansions for F_η, G, T_W etc. change sign on account of the sign of s. In the case of the surface temperature T_W, for example, Eqs. (21) give the change in T_W as being equal for compression and expansion waves to the first order in s, except for the difference in sign. The differences in $|T_W - T_0|$ thus don't appear until terms in s^2 are included.

The major difference in surface heat transfer and skin friction arises from the difference in density or pressure levels. For equal values of $|s|$ and x i.e., at equal distances x from the two wave heads when the waveheads are at equal distances $|a_0 t|$ from their respective focal points, the higher density of the compression-wave flow tends to have a dominant effect of causing a thinner boundary layer with larger normal gradients than the expansion-wave flow. This tendency is evident from the $y \to \eta$ compressibility transformation $\partial/\partial y = (\rho/\rho_0)[a_0/(v_0 x)]^{1/2}\, \partial/\partial\eta$ i.e., y gradients are proportional to η gradients multipled by the density.

For the limiting case of $K = \infty$, the compression-wave solution can thus be constructed from the corresponding expansion-wave solution for T_W constant given in Ref. 8. The expressions for the dimensionless skin-friction $\bar{\tau}_W$ and surface heat-transfer $\bar{q}_W$, for example, become

$$\bar{\tau}_W = \frac{\tau_W}{\rho_0 a_0}\left(\frac{|t|}{v_0}\right)^{1/2} = \pm\beta\frac{p_e}{p_0}|s|^{1/2} \times (F_0'' \pm F_1''|s| + F_2''s^2 + \ldots)_{\eta=0} \tag{22a}$$

$$\bar{q}_W = \frac{q_W}{\rho_0 a_0{}^2}\left(\frac{|t|}{v_0}\right)^{1/2} = \pm\frac{\beta p_e}{\sigma p_0}|s|^{1/2}\left(1 \mp \frac{\varepsilon|s|}{2}\right) \times (G_0' \pm G_1'|s| + G_2's^2 + \ldots)_{\eta=0} \tag{22b}$$

where the upper and lower signs apply, respectively, to expansion and compression flows and $|s|$ denotes the absolute value of s. The values of $F_0''(0)$ and $G_0'(0)$ are $2/\pi^{1/2}$ and $2(\sigma/\pi)^{1/2}$, respectively. The values of $F_1''(0)$, $F_2''(0)$, $G_1'(0)$ and $G_2'(0)$ from Ref. 8, are, respectively, 0.7946, 0.418, 0.1507, and -0.096 for $K = \infty$, $\gamma = 1.4$, and $\sigma = 0.72$. For these numerical values then, the preceding expressions for $K = \infty$ become

$$\bar{\tau}_W = \pm|s|^{1/2}\left(1 \mp \frac{|s|}{6}\right)^7 (0.9403 \pm 0.6622|s| + 0.348s^2 + \ldots) \tag{23a}$$

$$\bar{q}_W = \pm|s|^{1/2}\left(1 \mp \frac{|s|}{12}\right)\left(1 \mp \frac{|s|}{6}\right)^7 \times (1.108 \pm 0.1744|s| - 0.11s^2 + \ldots) \tag{23b}$$

Equations (23) are plotted in Fig. 4. The significant effect of the pressure (density), which appears in Eq. (23) as the seventh power factor, is evident in the strongly increased values of $\bar{\tau}_W$ and $\bar{q}_W$ for the compression wave relative to the expansion wave as $|s|$ increases.

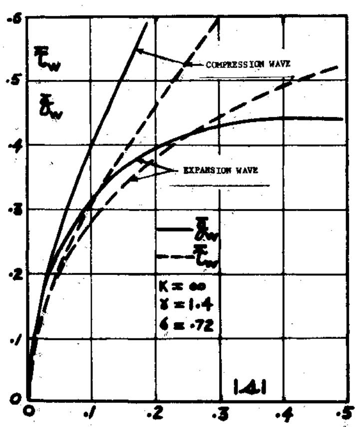

Fig. 4 Dimensionless skin friction and surface heat-transfer rate from Eq. (23) for compression and expansion waves.

Results for Normal Velocity v_e

In the next section the influence of the laminar wall boundary layer on ideal centered wave flow in a tube is analyzed in terms of the normal velocity v_e produced by the boundary layer at its outer edge. The required value of v_e may be obtained via Eq. (1), i.e.,

$$v_e = -1/\rho_e \int_0^\infty [\partial\rho/\partial t + (\partial/\partial x)(\rho\bar{u})]\, dy$$

with application of the various transformations used. The final result may be expressed as follows:

$$v_e/a_0 s(a_0 x/v_0)^{1/2} = v_e(t/v_0 s)^{1/2} = (p_e/p_0)(A_0 + A_1 s + A_2 s^2 + \ldots) \tag{24a}$$

$$A_0 = \tfrac{3}{2}(1-\varepsilon)f_0 + \tfrac{3}{2}(b_1 + 2\varepsilon)g_0 \tag{24b}$$

$$A_1 = \tfrac{1}{2}\{(\varepsilon - 1)(6\varepsilon f_0 + 5f_1) + [(b_1 + 2\varepsilon)(3-\varepsilon) + 5(b_2 - \varepsilon^2)]g_0 - 5(b_1 + 2\varepsilon)g_1\} \tag{24c}$$

$$A_2 = \tfrac{1}{2}\{(1-\varepsilon)(3\varepsilon^2 f_0 + 10\varepsilon f_1 - 7f_2) + [3(1-\varepsilon)(b_2 - \varepsilon^2) + 7b_3]g_0 - [3(1-\varepsilon)(b_1 + 2\varepsilon) + 7(b_2 - \varepsilon^2)]g_1 - 7(b_1 + 2\varepsilon)g_2\} \tag{24d}$$

$$f_0 = \lim_{\eta\to\infty}[\eta - F_0(\eta)] = 0.7509 \tag{24e}$$

$$f_1 = F_1(\infty) = f_1(\gamma, \sigma) \tag{24f}$$

$$f_2 = F_2(\infty) = f_2(\gamma, \sigma) \tag{24g}$$

$$g_0 = \lim_{\eta\to\infty}\left[\eta - \int_0 G_0\, d\eta\right] = f_0/\sigma^{1/2} \tag{24h}$$

$$g_1 = \int_0^\infty G_1\, d\eta = g_1(\gamma, \sigma) \tag{24i}$$

$$g_2 = \int_0^\infty G_2\, d\eta = g_2(\gamma, \sigma) \tag{24j}$$

It is noted that the sign of v_e from Eq. (24) depends on the sign of s. Thus v_e is positive, or directed away from the wall, for the expansion wave boundary layer, and negative, or towards the wall, for the compression wave flow. It is also apparent that the absolute magnitude of v_e is increased by heat transfer in both cases. The higher stream pressure of the compression wave flow has the effect of somewhat diminishing v_e compared to the expansion flow case.

For considerations of the boundary-layer influence on the inner inviscid flowfield in a tube, the influence of the wall temperature change previously considered will typically be unimportant. If in that case the wall conductivity is assumed infinite and the b_i values are accordingly zero for $i \geqq 1$, the values for the remaining constants f_1 etc. in Eq. (24) can then be determined from the expansion-wave solution of Ref. 8. Those values are $f_1 = 0.567$, $f_2 = -0.114$, $g_1 = 0.437$, and $g_2 = -0.026$, for $\gamma = 1.4$ and $\sigma = 0.72$. The corresponding values of the A coefficients in Eq. (24) then become $A_0 = 1.379$, $A_1 = -1.504$, $A_2 = 0.6125$. Thus the magnitude of v_e/a_0 is of the order of $s/(a_0 x/v_0)^{1/2}$.

Displacement Effect of Tube Wall Boundary Layer

The influence of a thin boundary layer on the inviscid flow in a tube can be analyzed in terms of the normal velocity v_e which exists at the outer edge of the boundary layer. This approach has been well established by Mirels[6] for quasi-steady shock-tube boundary layers. In the present problem, the boundary layer influence will differ for expansion- and compression-wave flows because of the difference noted in the direction of v_e. The effect of v_e will be to generate compressive disturbances in the expansion-wave flow and expansive disturbances in the compression-wave flow.

In order to estimate the quantitative effects of v_e, the flow is assumed to be generated in a tube of constant hydraulic diameter D. The wall boundary-layer thickness is considered sufficiently smaller than D so that the interior flow can be assumed isentropic and spatially one-dimensional, with velocity $u(x', t')$ in tube-fixed coordinates. It can then be shown that to a first approximation the flow is governed by the following equations.

Mass

$$\partial\rho/\partial t' + (\partial/\partial x')(\rho u) = (4/D)\rho v_e \tag{25a}$$

Momentum

$$\partial u/\partial t' + u\,\partial u/\partial x' = -(1/\rho)\,\partial p/\partial x' \tag{25b}$$

Entropy

$$p/p_0 = (\rho/\rho_0)^\gamma = (T/T_0)^{\gamma/(\gamma-1)} \tag{25c}$$

The boundary-layer presence is thus felt only through a mass source or sink term proportional to v_e/D.

It is convenient to utilize the relations between u and sound speed a which apply along the two families of characteristic curves I and II for Eqs. (25). The characteristic relations derived from Eqs. (25) are

$$(dx'/dt')_{\rm I} = u+a, \qquad [d\Delta u+(2/\gamma-1)\,d\Delta a]_{\rm I} = [(4av_e/D)\,dt']_{\rm I} \tag{26}$$

$$(dx'/dt')_{\rm II} = u-a, \qquad [d\Delta u-(2/\gamma-1)\,d\Delta a]_{\rm II} = -[(4av_e/D)\,dt']_{\rm II} \tag{27}$$

where Δu and Δa are the changes due to the boundary-layer influence, i.e., $u = u_e+\Delta u$ and $a = a_e+\Delta a$.

The changes Δu and Δa at any point A with coordinates s, t, lying within the wave region may now be obtained by appropriate integration of Eqs. (26) and (27) along the corresponding characteristic paths I and II which influence conditions at A. Figures 1 and 2 illustrate the integration paths involved for expansion and compression waves, respectively. For the expansion wave, Fig. 1, the boundary conditions are $\Delta u = \Delta a = 0$ at point B and at $t' = 0$. It then follows that for the expansion wave

$$\Delta a = \frac{\gamma-1}{4}\left[\underset{\rm I}{\int_B^A f\,dt'} + \underset{\rm II}{\int_O^A f\,dt'}\right]$$

$$\Delta u = \frac{1}{2}\left[\underset{\rm I}{\int_B^A f\,dt'} - \underset{\rm II}{\int_O^A f\,dt'}\right]$$

where $f = 4av_e/D$. For the compression wave, Fig. 2, the boundary conditions are $\Delta u = \Delta a = 0$ at points B and D, and $\Delta u = 0$ at point C on the piston path. The vanishing of Δu at C is required because the piston path is prescribed a priori by the relation

$$t = t'-t_0' = -t_0'(1-\varepsilon s)^{-1/\varepsilon}$$

which gives a wave focused at t_0' in the absence of boundary-layer effects. The corresponding expressions for the compression wave case are then

$$\Delta a = \frac{\gamma-1}{4}\left[\underset{\rm I}{\int_B^A f\,dt'} + \underset{\rm II}{\int_C^A f\,dt'} + \underset{\rm I}{\int_D^C f\,dt'}\right]$$

$$\Delta u = \frac{1}{2}\left[\underset{\rm I}{\int_B^A f\,dt'} - \underset{\rm II}{\int_C^A f\,dt'} - \underset{\rm I}{\int_D^C f\,dt'}\right]$$

In the preceding integrals, v_e may be prescribed for either a laminar or a turbulent boundary layer. Because of the boundary-layer influence, the integration paths are distorted from the inviscid characteristic curves which obtain without the boundary-layer present. Although the path distortions can be simultaneously calculated, for example by suitable expansions or with numerical methods, the present effort was limited to the approximation of neglecting the path distortions and evaluating the integrals along the inviscid characteristic curves. This procedure is equivalent to linearization and provides linearized first approximations for Δu and Δa. The inviscid characteristic curves have the simple forms $t = t_{B,D}(1-\varepsilon s)^{-1/(2\varepsilon)}$ for curve I through points B or D, and $s = s_A =$ constant for curve II through A.

With v_e prescribed as the normal velocity for the laminar boundary layer given by Eq. (24) the integrations along the inviscid characteristic paths are simple. The final results for Δu and Δa for both compression and expansion waves may be summarized as follows.

$$\frac{\Delta u}{a_0} = \frac{4}{(a_0 D/v_0)^{1/2}}\,\frac{(a_0\,st/D)^{1/2}}{p_e/p_0}\,[\phi-\psi]\left[1-\left(\frac{t_c}{t}\right)^{1/2}\right] \tag{28a}$$

$$\frac{\Delta a}{a_0} = \frac{2(\gamma-1)}{(a_0 D/v_0)^{1/2}}\,\frac{(a_0\,st/D)^{1/2}}{p_e/p_0}\left[\phi+\psi+(\phi-\psi)\left(\frac{t_c}{t}\right)^{1/2}\right] \tag{28b}$$

$$t_c = -t_0'(1-\varepsilon s)^{-1/\varepsilon} \tag{28c}$$

$$\phi = \phi(s) = \frac{A_0}{6}\,s + \frac{1}{10}\left(A_1 - \frac{5+4\varepsilon}{6}\right)s^2 \tag{28d}$$

$$\psi = \psi(s) = A_0+(A_1-\varepsilon A_0)s+(A_2-\varepsilon A_1)s^2 \tag{28e}$$

The constants, A_0, A_1, A_2 are the coefficients for v_e as given in the preceding section. The time t_c, corresponding to point C on the piston path of Fig. 2, depends on s and t_0' as given previously. The ratio t_c/t is $\geqq 1$ for the compression wave and zero for the expansion wave (as t_0' is then zero).

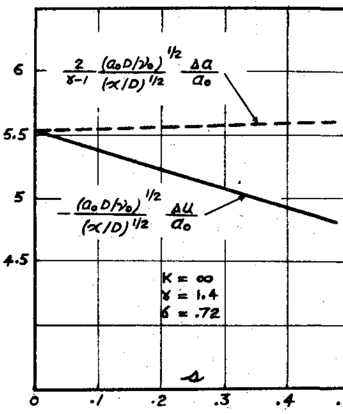

Fig. 5 Inviscid tube flow perturbations due to displacement effect of expansion-wave boundary layer, from Eq. (28).

The solutions given by Eq. (28) show the behavior expected according to the sign of v_e: for expansion waves $\Delta a > 0$ and heating occurs, and for compression waves $\Delta a < 0$ and cooling occurs. In both cases the magnitude of the flow velocity is reduced, i.e., $|u_e+\Delta u| < |u_e|$. Thus the effect of the boundary layer is to reduce or diminish the basic changes in u, T, p etc. produced by either wave. Graphical results from Eq. (28) are shown in Fig. 5 for the particular case of the expansion-wave flow ($t_0' = 0$). The perturbation quantities plotted are seen to be relatively insensitive to s, so that $\Delta u/a_0$ and $\Delta a/a_0$ vary mainly as $[(x/D)/(a_0\,D/v_0)]^{1/2}$. The perturbations are typically quite small, $\Delta u/a_0$ and $\Delta a/a_0$ being of the order of a few percent or less for x/D in the range 1–100 and the Reynolds number $a_0\,D/v_0$ in the range 10^6 to 10^7. The main contributions to Δu and Δa come from the integrals along path II ($s =$ const) which are proportional to the funtion ψ. The integrals along path I are proportional to ϕ which is an order smaller than ψ.

The solutions for the corresponding effects of the boundary layer on the inviscid flow pressure and temperature, p and T, are given in terms of Δa by the linearized isentropic relationships which are as follows.

$$\begin{aligned}\Delta T/T_0 &= 2(1-\varepsilon s)\,\Delta a/a_0+\ldots \\ \Delta p/p_0 &= (1+\varepsilon)/(\varepsilon)\,(1-\varepsilon s)^{1/\varepsilon}\,\Delta a/a_0+\ldots\end{aligned} \tag{29}$$

The pressure perturbation Δp has, of course, the same sign as Δa, the pressure being increased by boundary-layer effects in the expansion flow and decreased in the compression flow.

Conclusions

This study considered the laminar boundary layer produced by centered unsteady waves moving into gas at rest, and the resultant effects of the boundary layer on the inviscid wave flow in a tube. By describing the streamwise variations in terms of distance measured relative to the wave head and time measured relative to the wave focal point, a unified formulation is obtained which provides boundary-layer solutions for both compression and expansion wave flows. The effects due to the variation in wall surface temperature which occurs with finite thermal conductivity of the wall have been included.

The results for wall temperature variation indicate that in some experimental applications of such unsteady waves at high pressures, the wall temperature variation significantly reduces the surface heat-transfer rate. The displacement effect of the boundary layer on the inviscid tube flow depends on the direction of the normal velocity v_e at the boundary-layer edge, which differs for expansion and compression waves. However, for either type of wave the net effect of the boundary layer is to diminish the basic changes in velocity, temperature, etc. which the wave produces. The magnitude of the inviscid flow perturbation is typically small, being of the order of

$$[(x/D)/(a_0\,D/v_0)]^{1/2}$$

In the integration along the characteristic curves to determine the inviscid flow perturbations, the distortion of the characteristic curves themselves by boundary-layer influence has been neglected, which is equivalent to a linearization. The distorted characteristic curves can be obtained to a linearized approximation using the solutions obtained for Δu and Δa, i.e., by integration of the changes in slope $\Delta u+\Delta a$ along I, and $\Delta u-\Delta a$ along II. This aspect has not been further pursued in the present study except to note that the quantity $|u-a|$, which determines the slope of the family II characteristics, is locally increased for expansion waves and decreased for compression waves by the boundary-layer influence. Thus in the x', t' plane the family II characteristics are curved by the boundary-layer influence, becoming concave upwards for compression waves and concave downwards for expansion waves. At any point x', t' within either wave the distance to the wave focal point thus appears to be increased as determined by the local slope of characteristic II. For the compression wave, for example, the qualitative effect is to delay the focusing of the wave.

Of course in practice, perfectly centered waves do not usually occur even with negligible boundary-layer effects. The present solutions, however, might reasonably be expected to give at least a qualitative indication of laminar boundary-layer effects in wave flows which are not centered. Finally, it might be emphasized that the present solutions are valid only for a laminar boundary layer, and for points lying within the wave regions and for which $|s|$ is not too large, say not exceeding about 0.3 to 0.4 at most. The error incurred is presumably of the order of s^2 or s^3 for the expansions to terms in s or s^2, respectively.

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